Lecture 19

Max-Flow Min-Cut Theorem

Source: Introduction to Algorithms, CLRS and Kleinberg & Tardos

Ford-Fulkerson Method: Correctness

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Examples:

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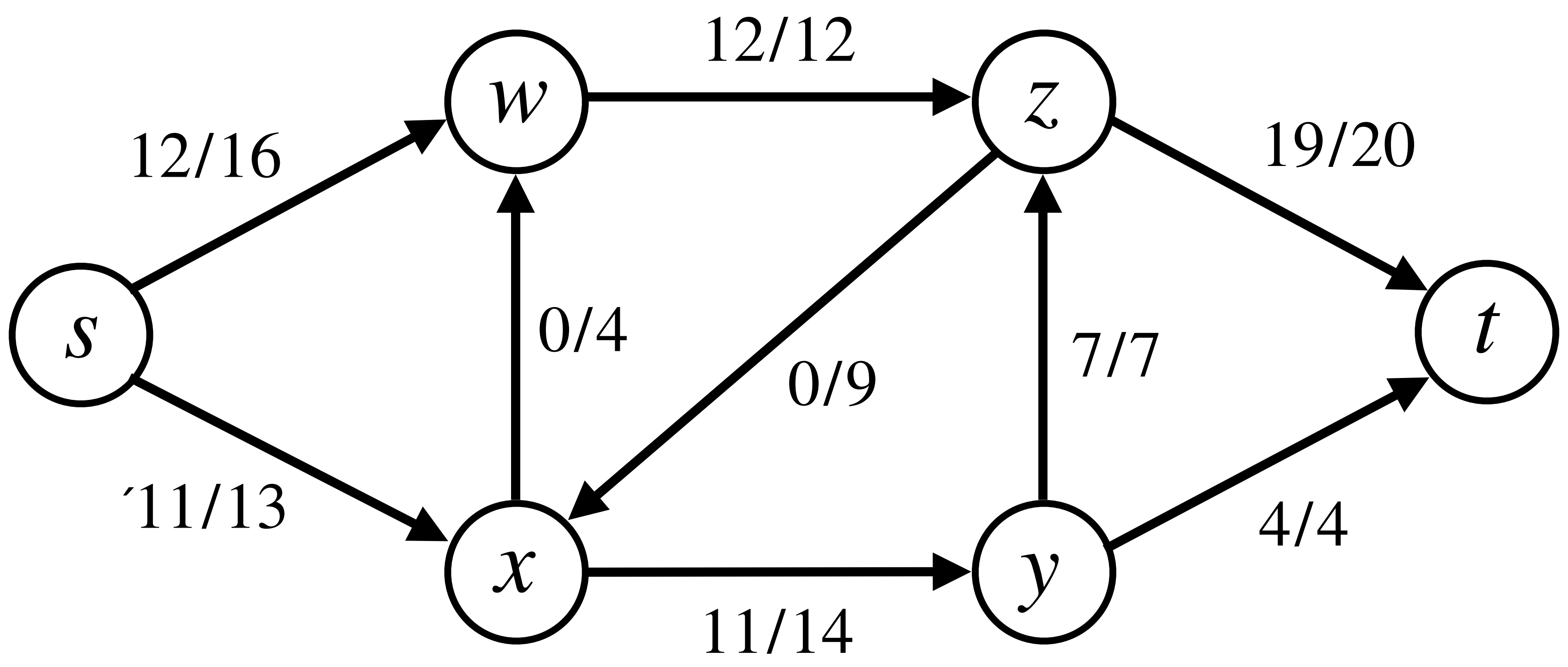

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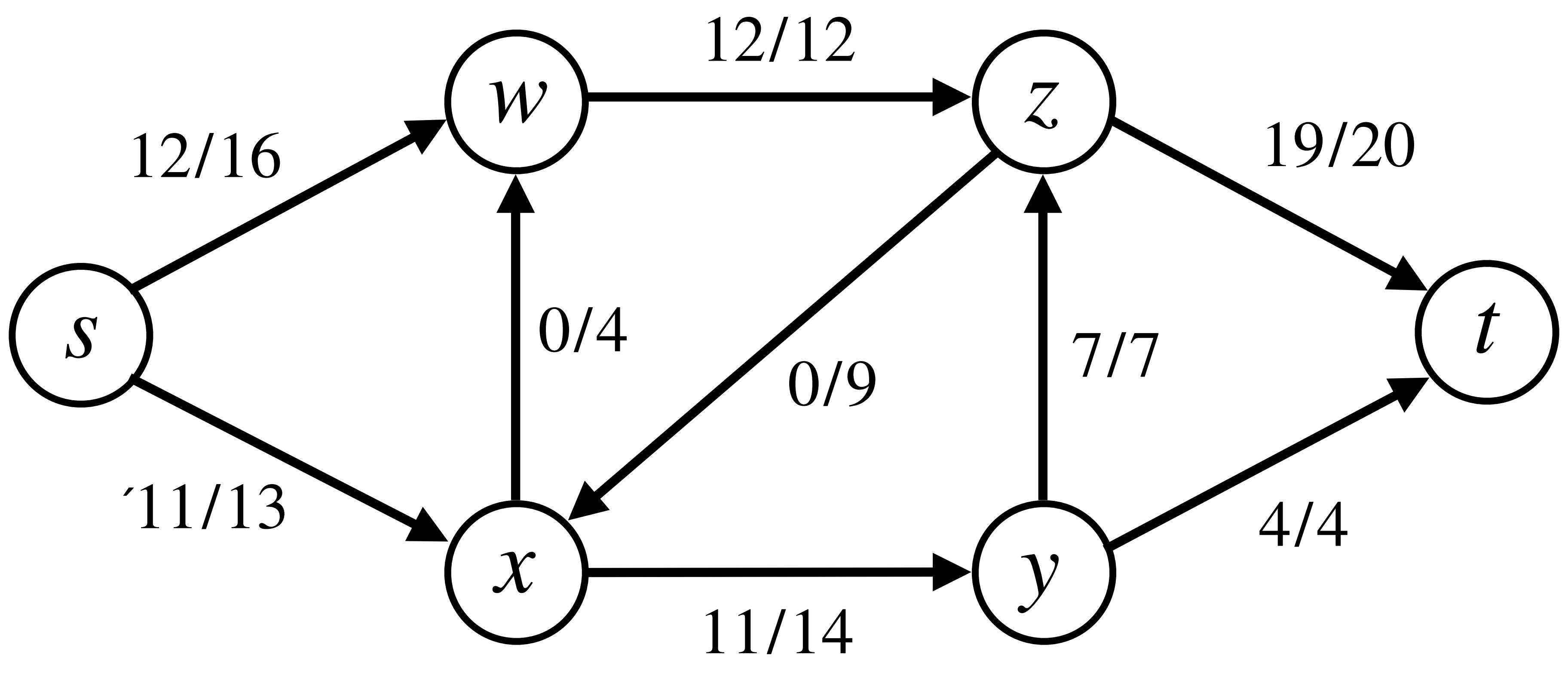

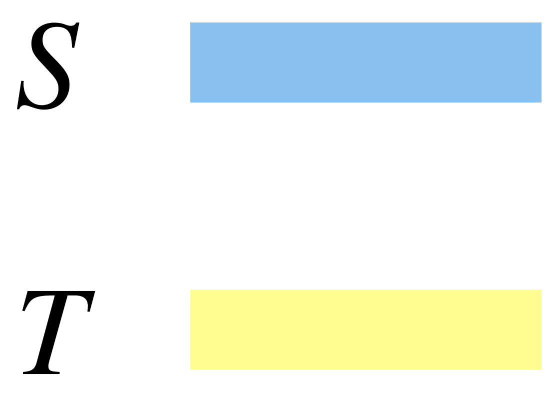

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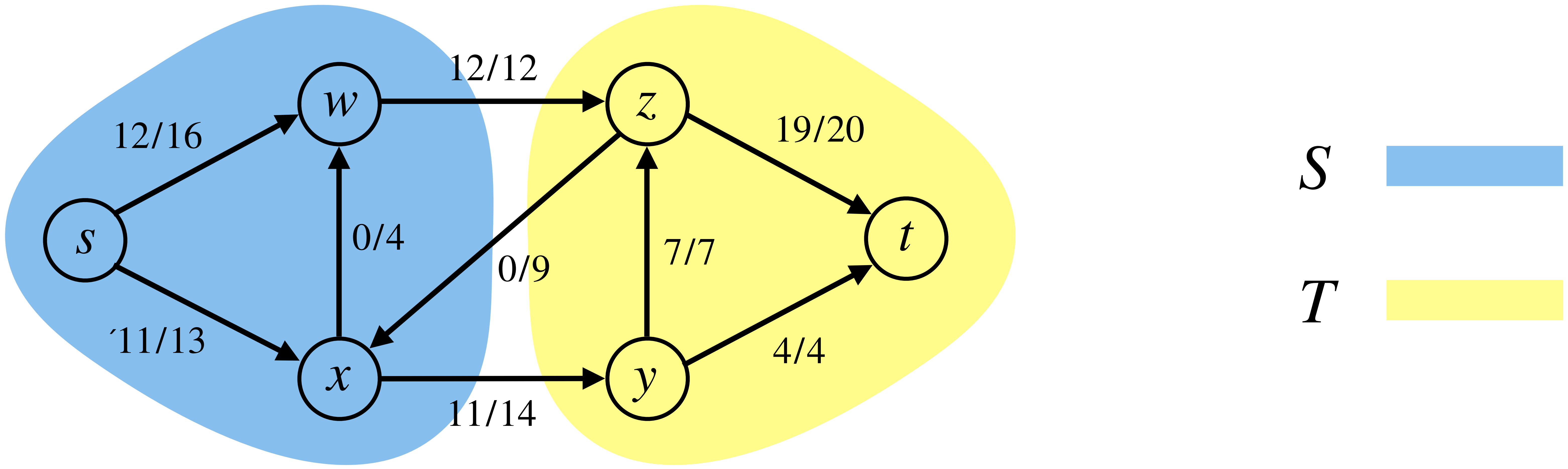

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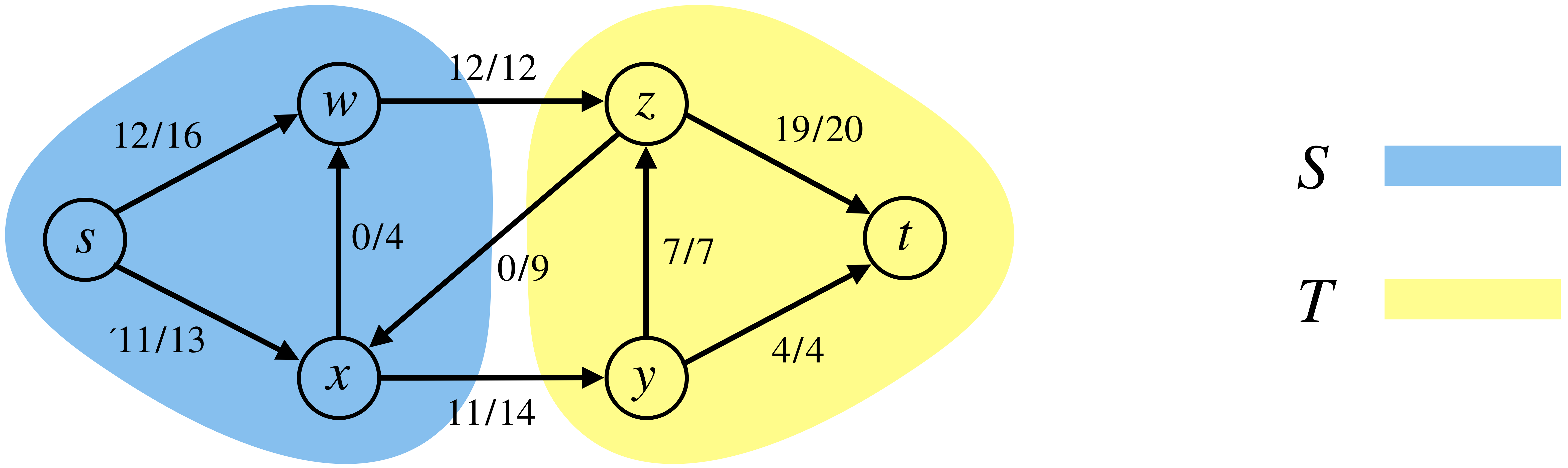
Examples:

$c(S, T) = 26$



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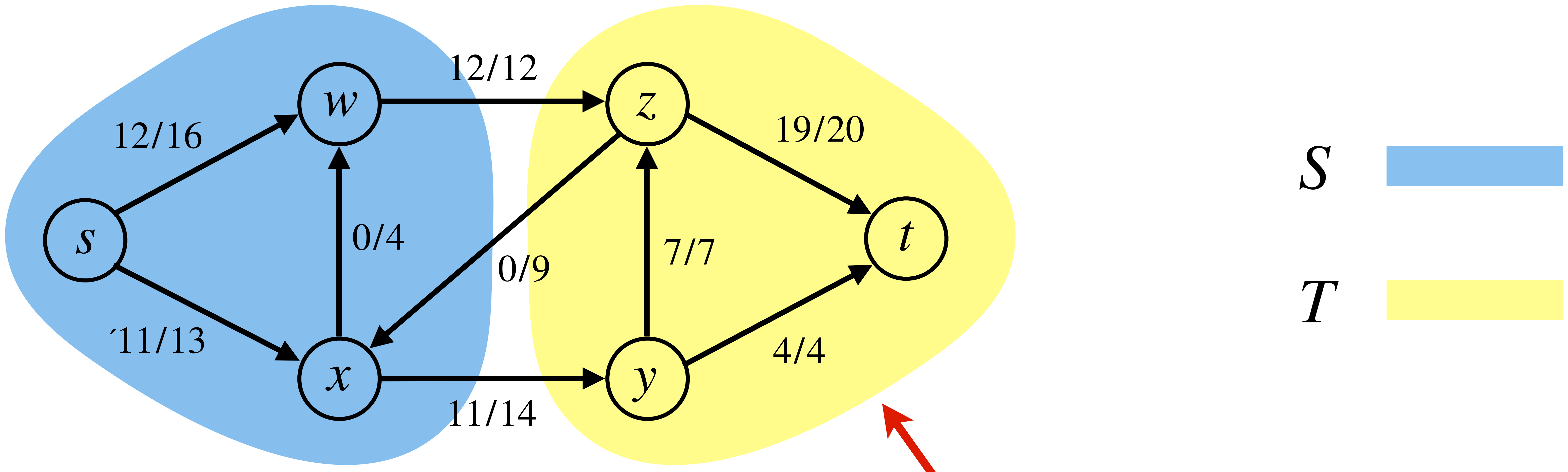

Is a flow of value 27 possible?

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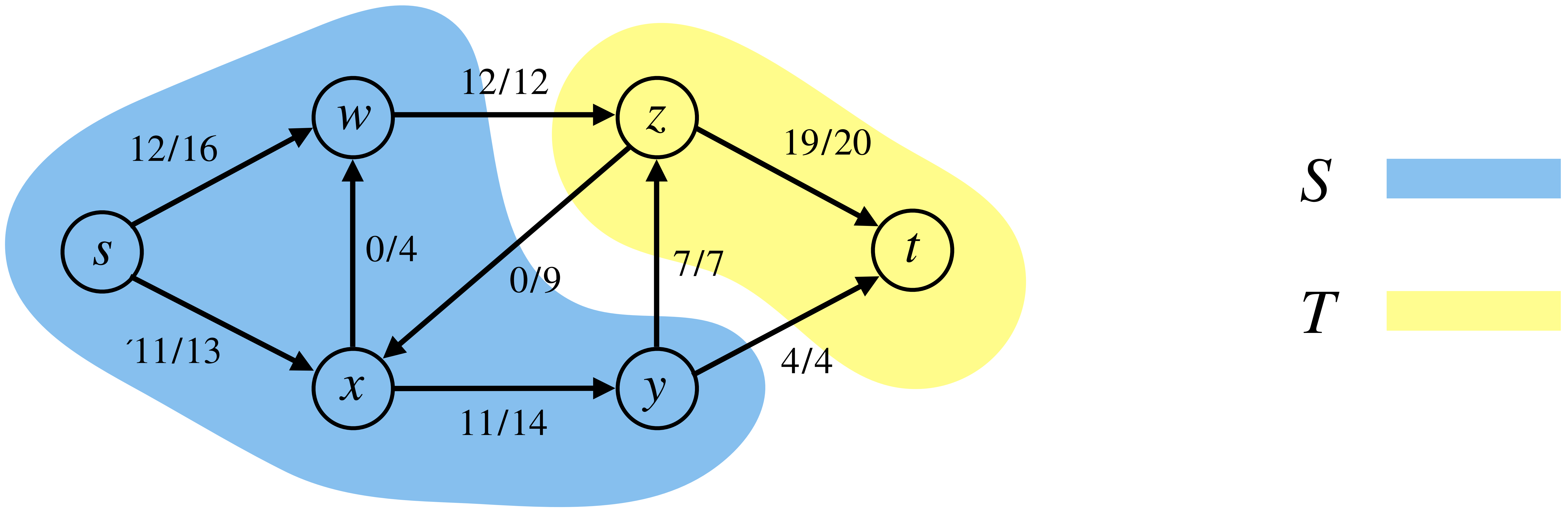

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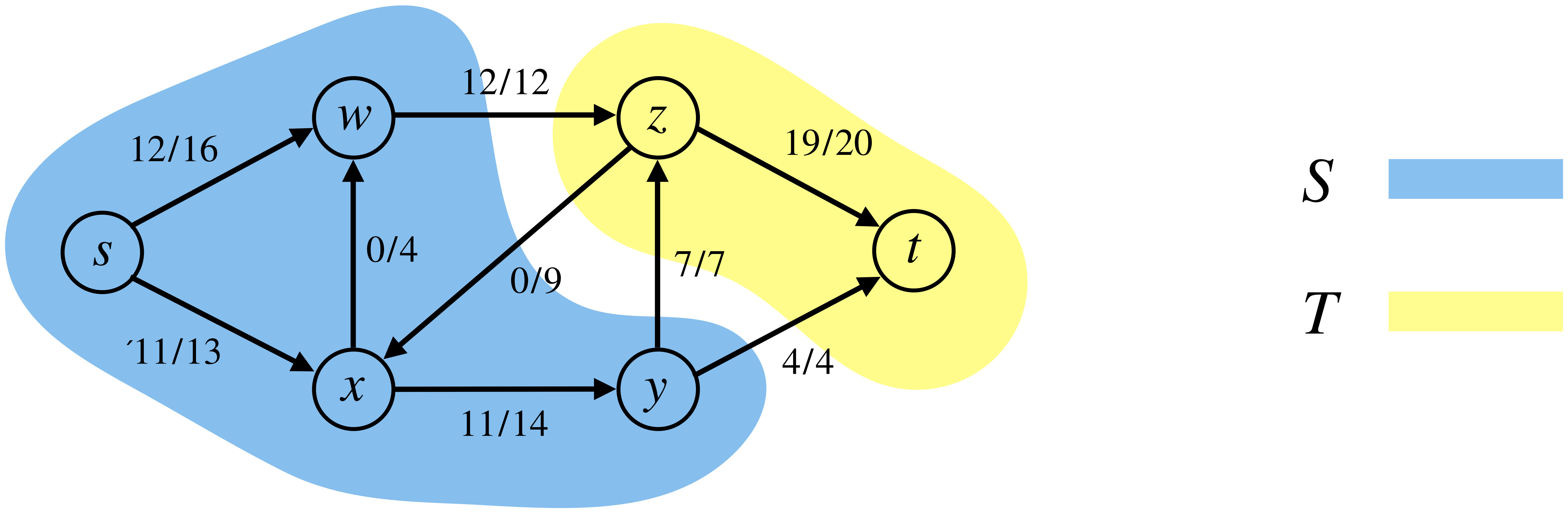

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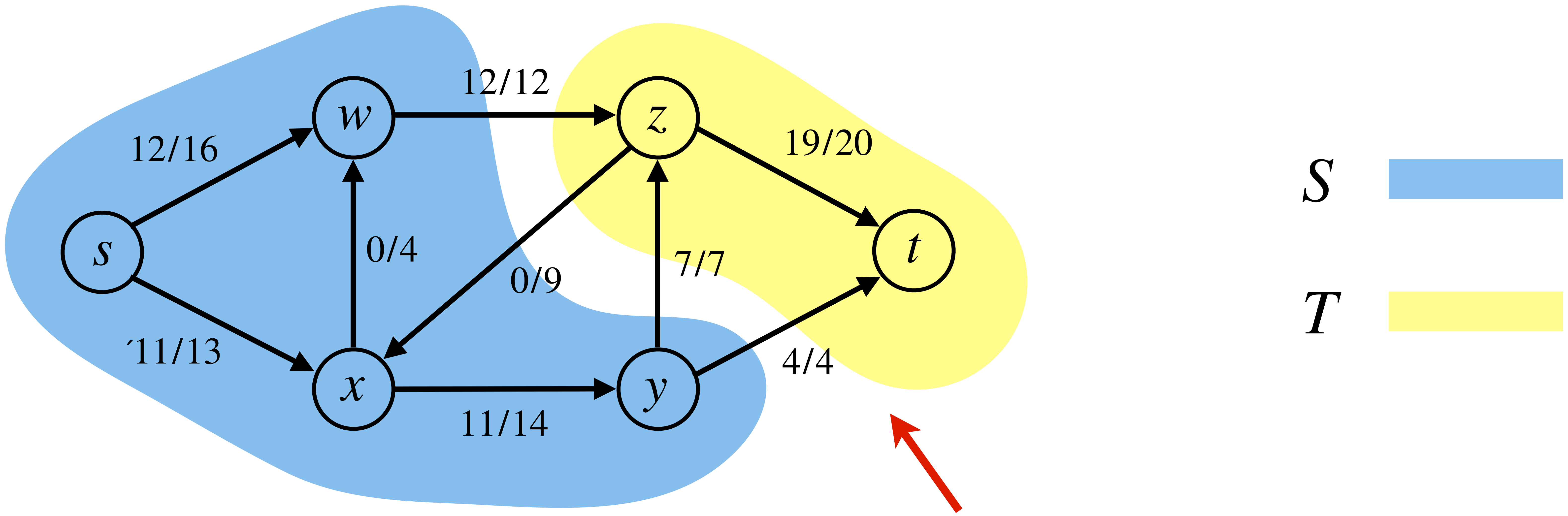

Is a flow of value 24 possible?

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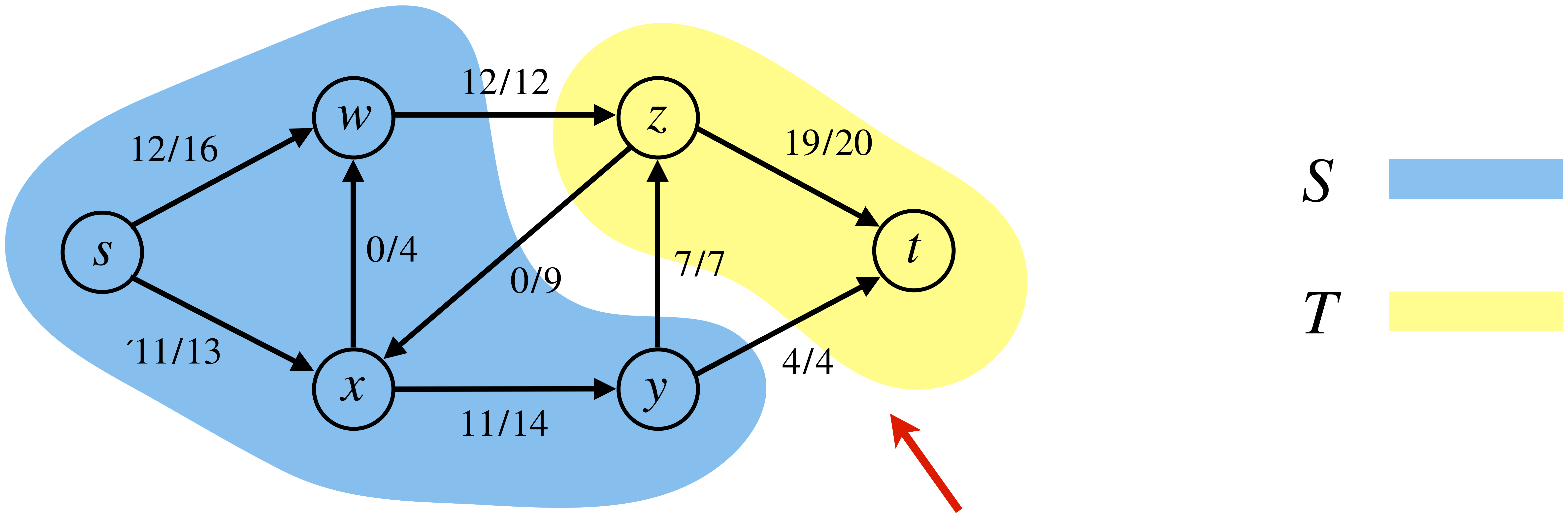

Capacity of minimum cut bounds the value of any flow

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↑

We just proved this

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Defn: The capacity of a cut (S, T) is defined as

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This is enough to prove Ford-Fulkerson method correct

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Such a cut must exist due to above theorem.

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↑ Value of any flow $\leq$ Capacity of any cut

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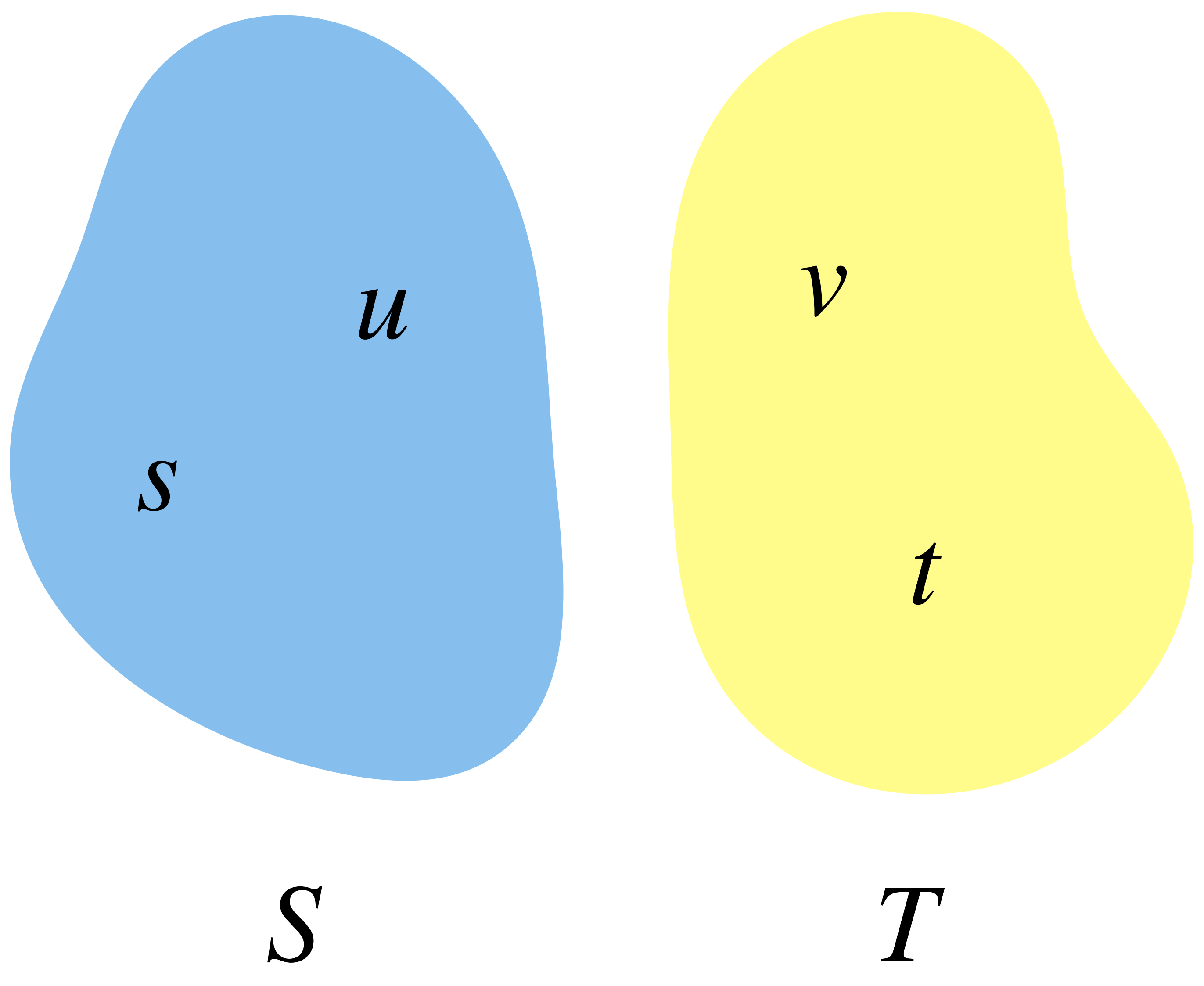

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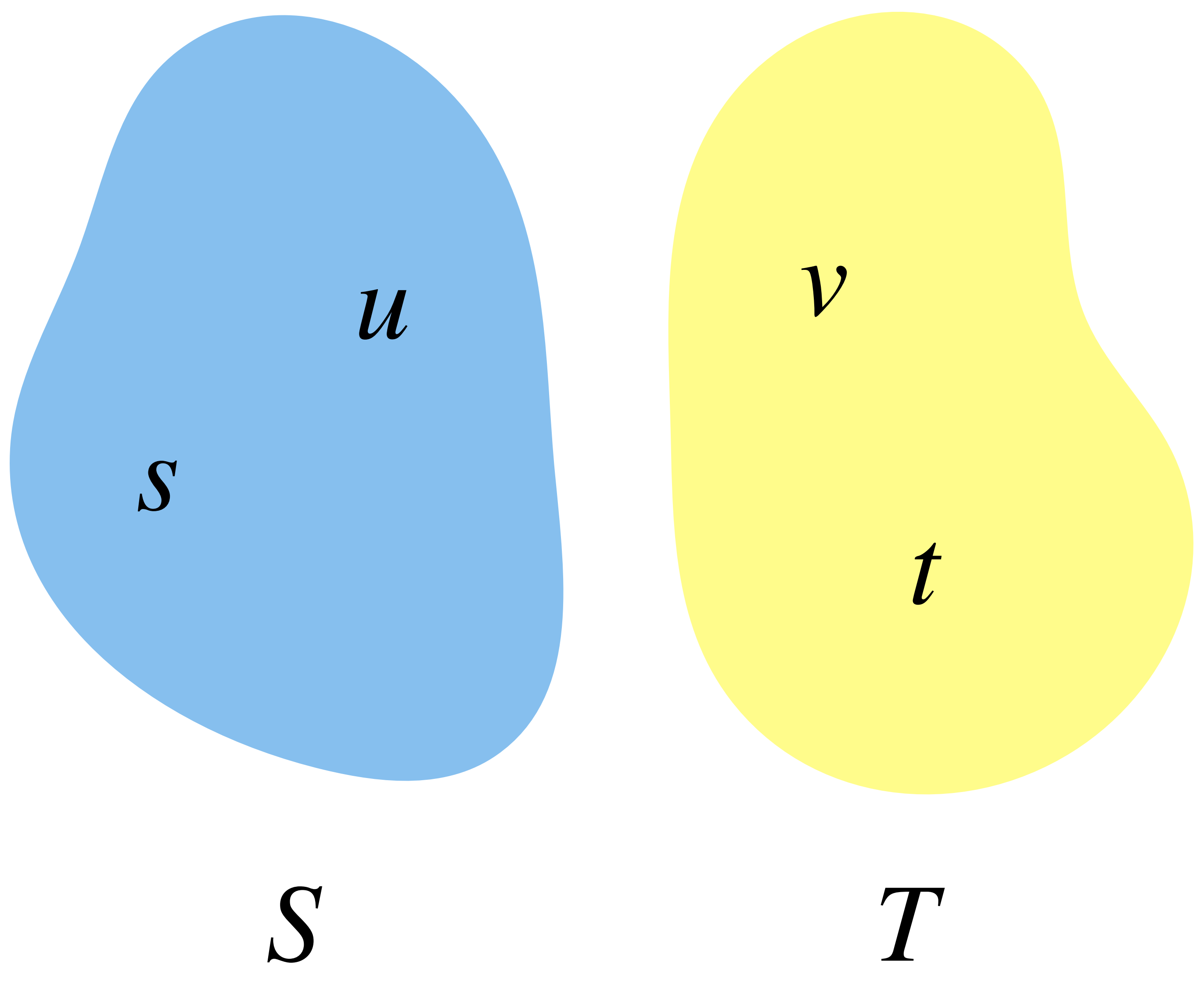

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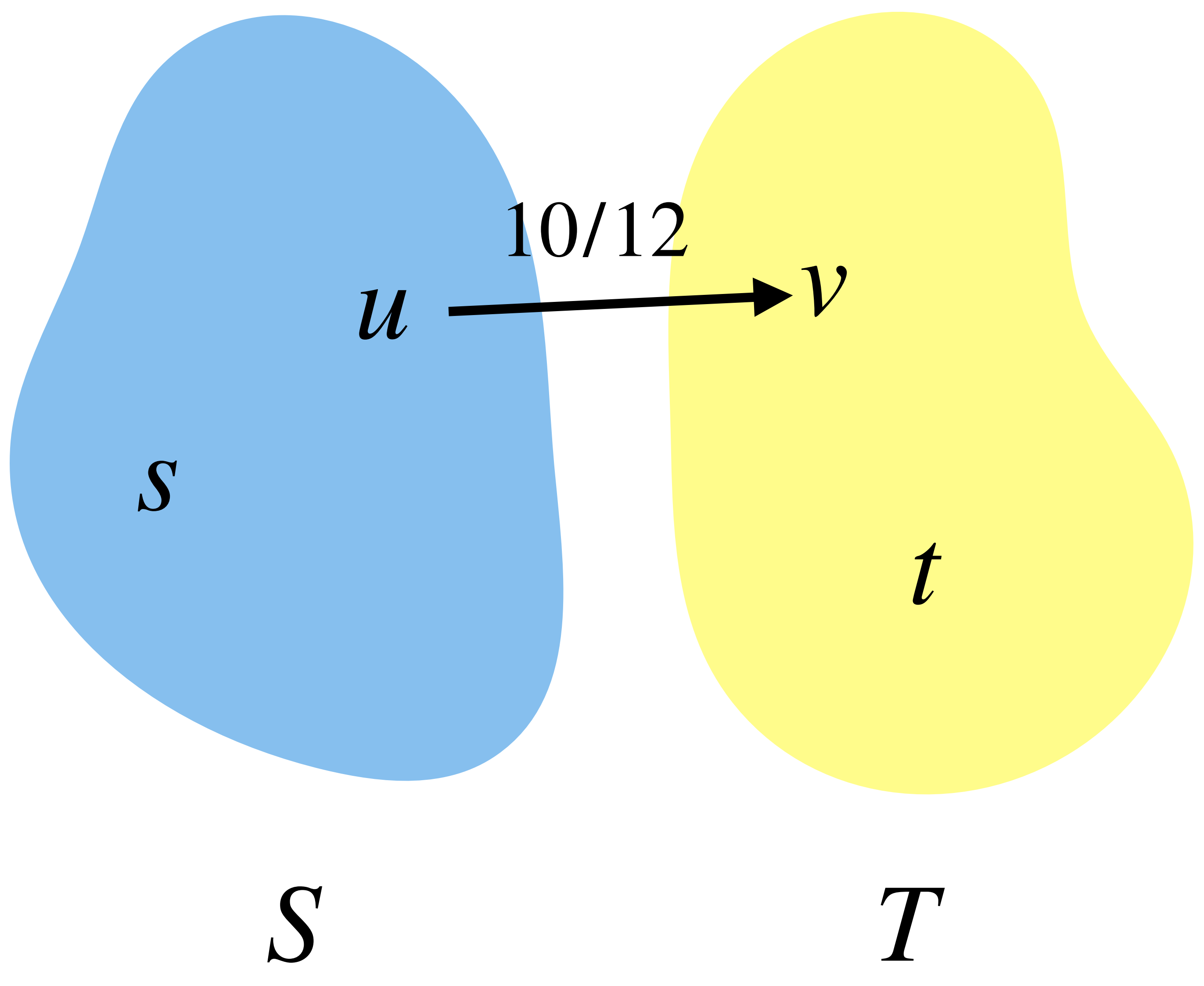

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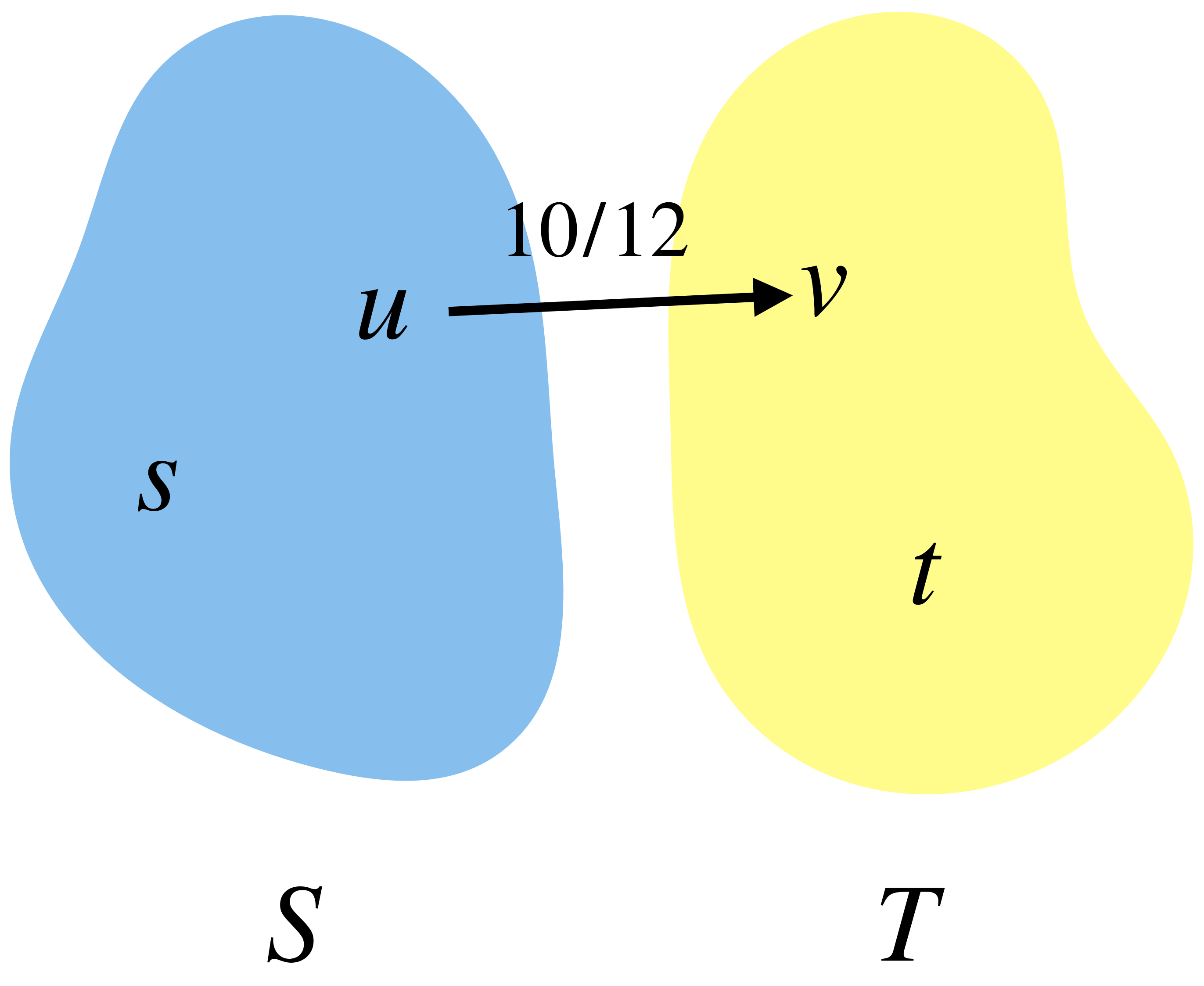

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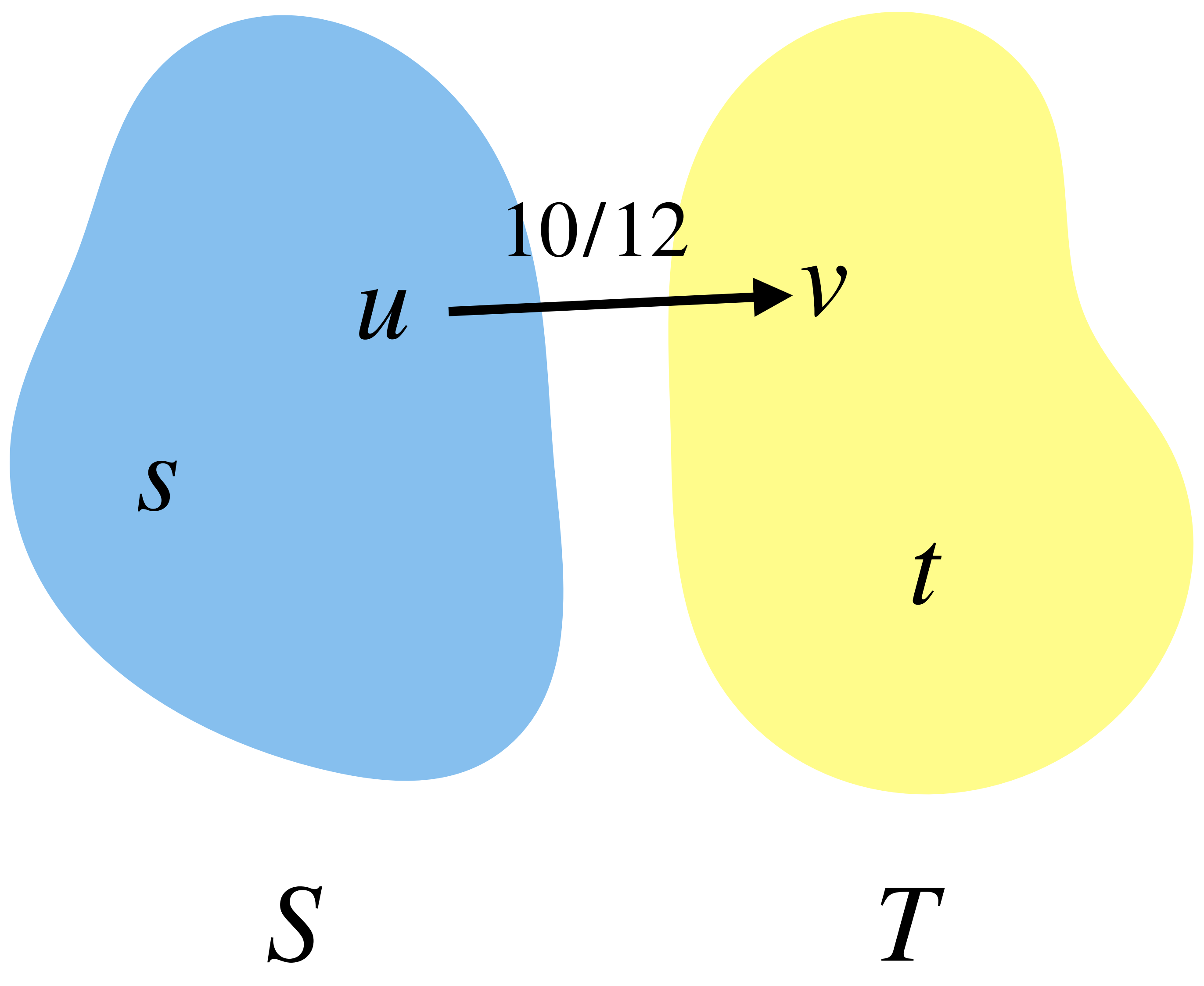

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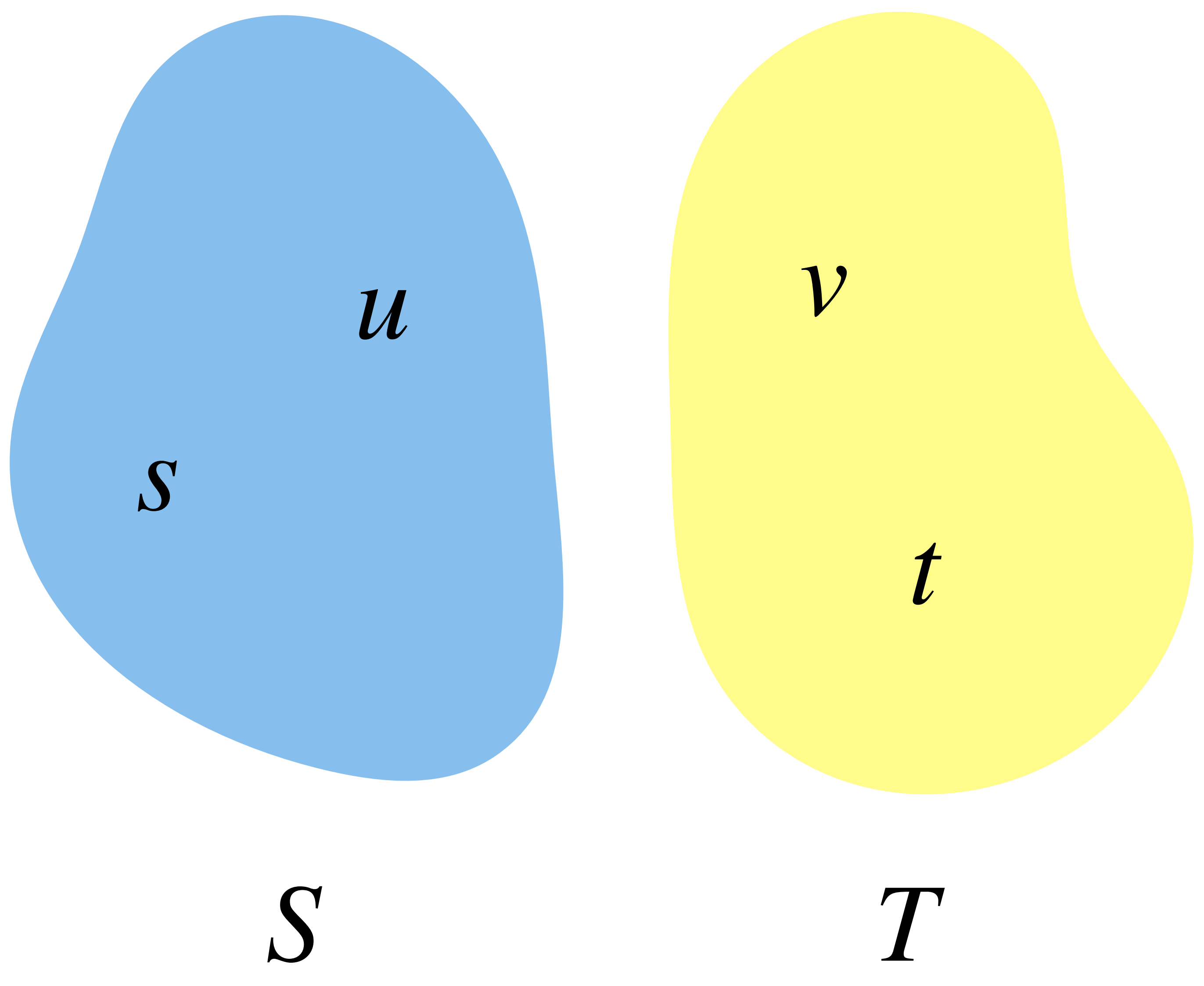

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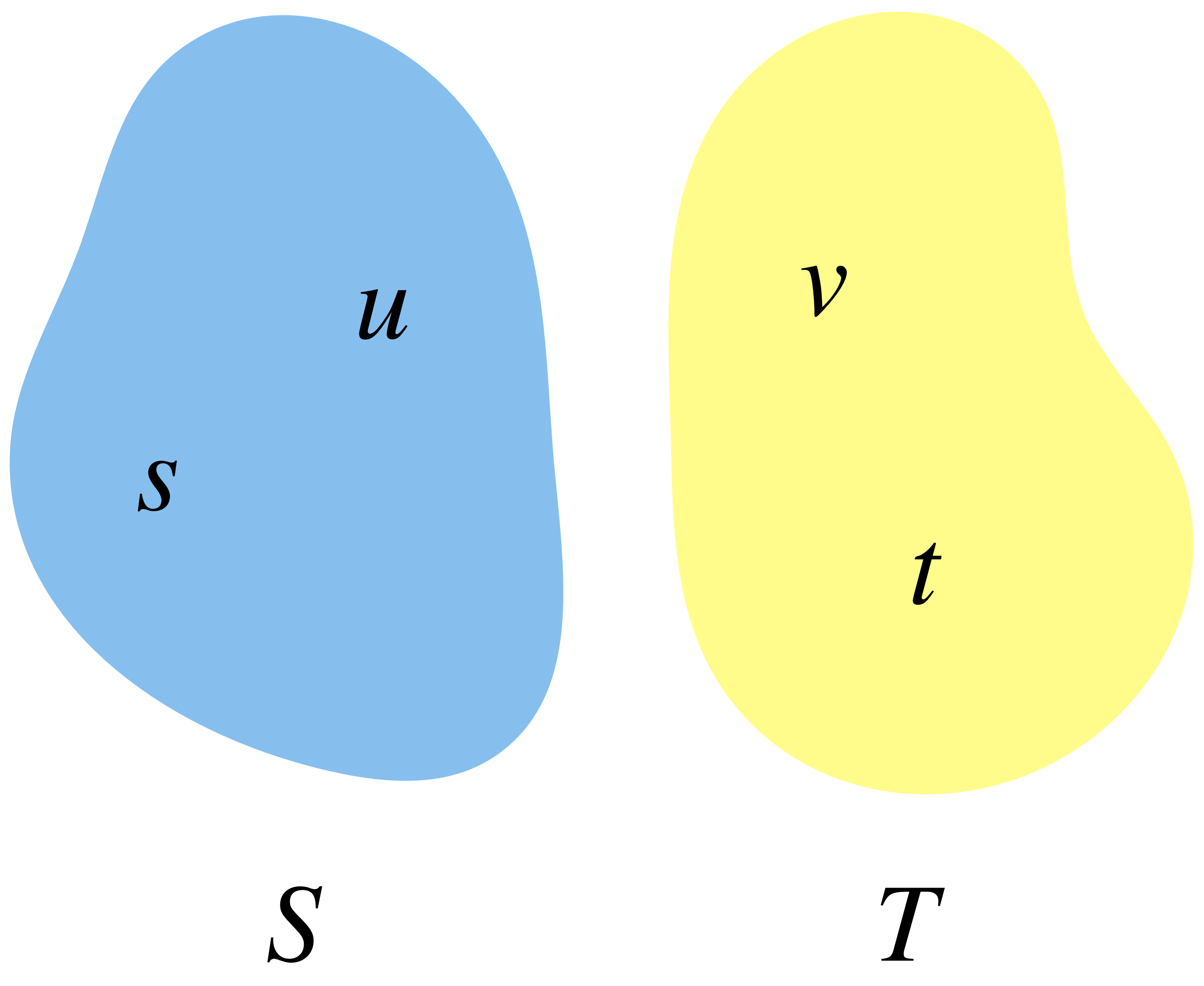

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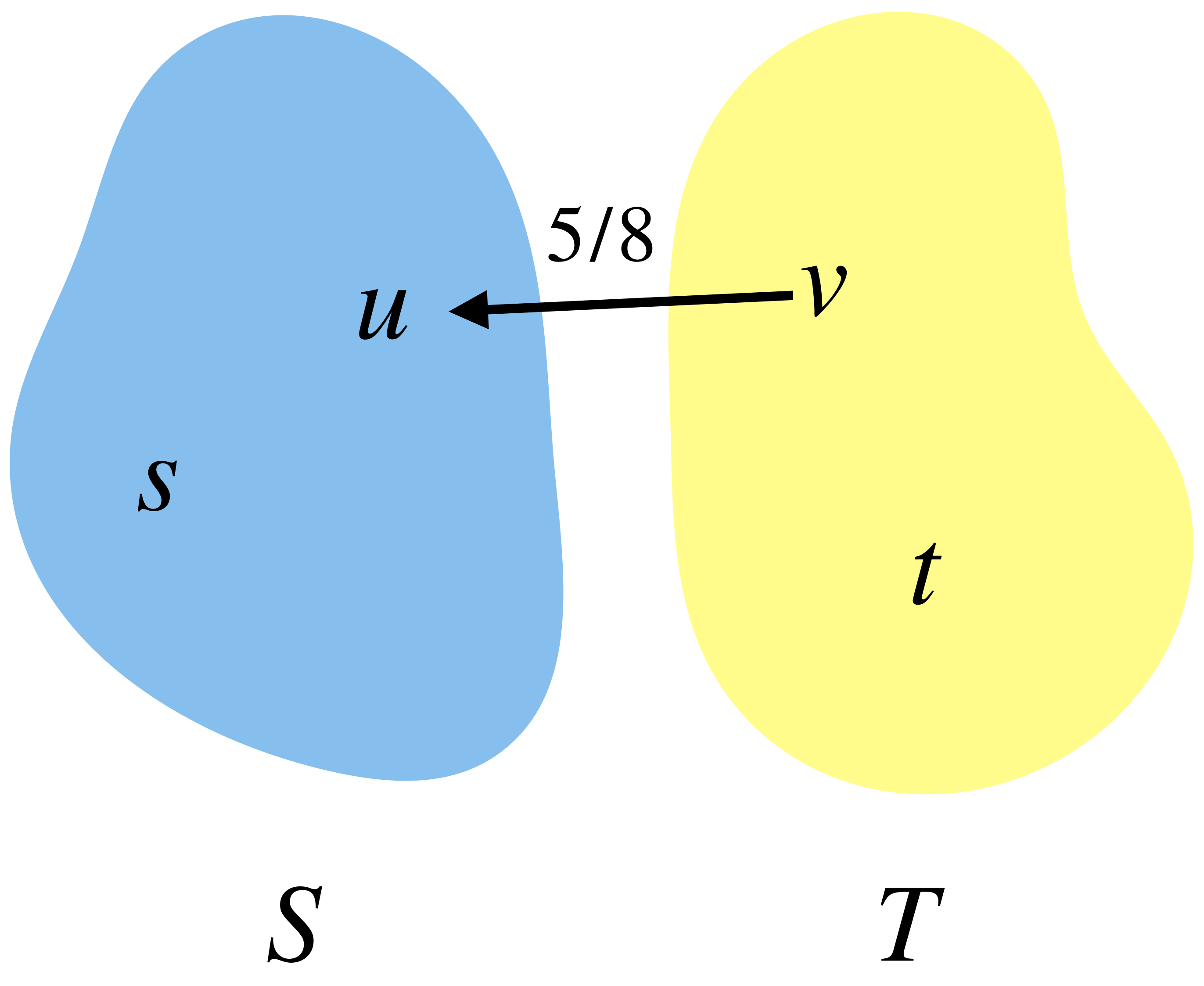

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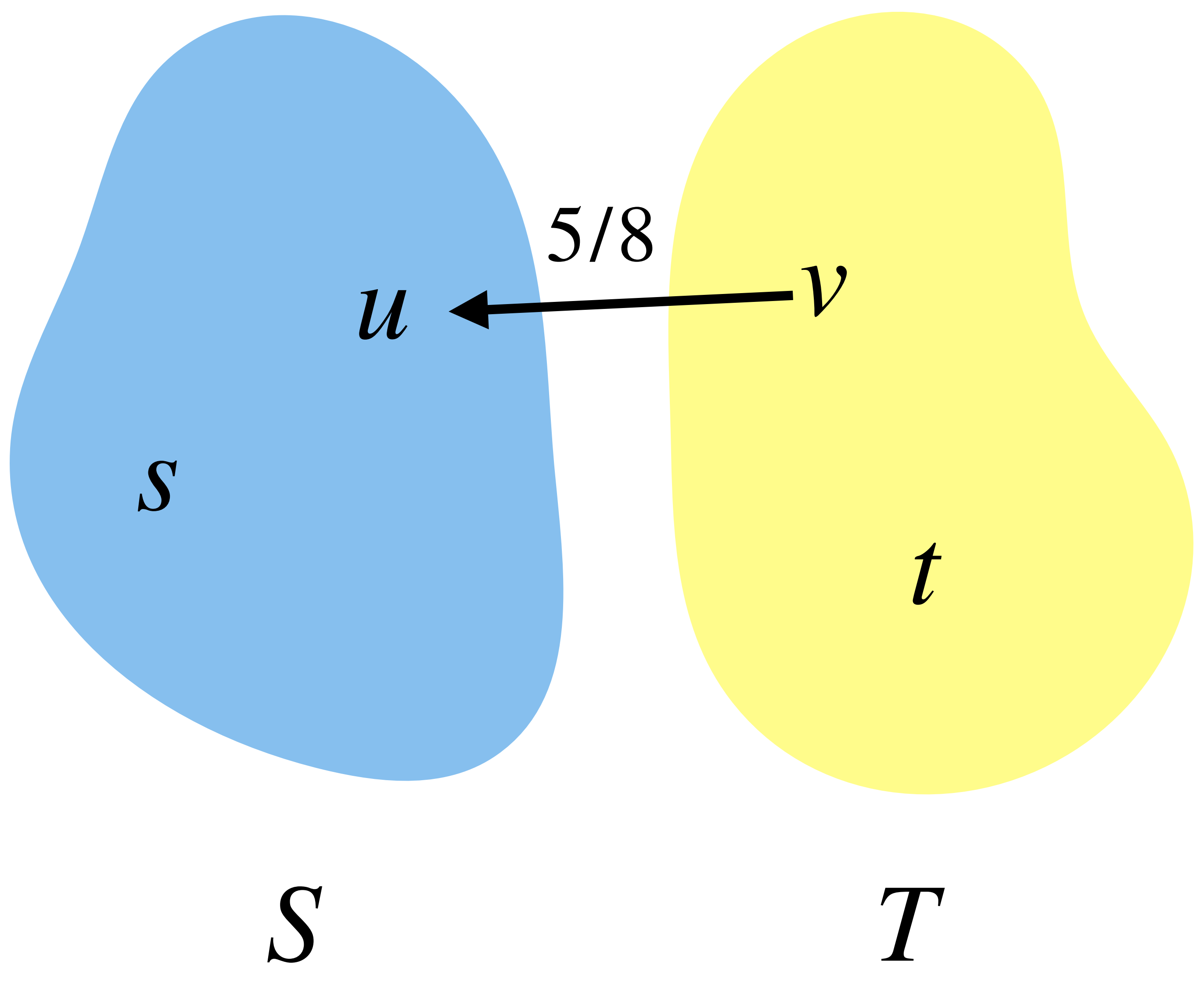

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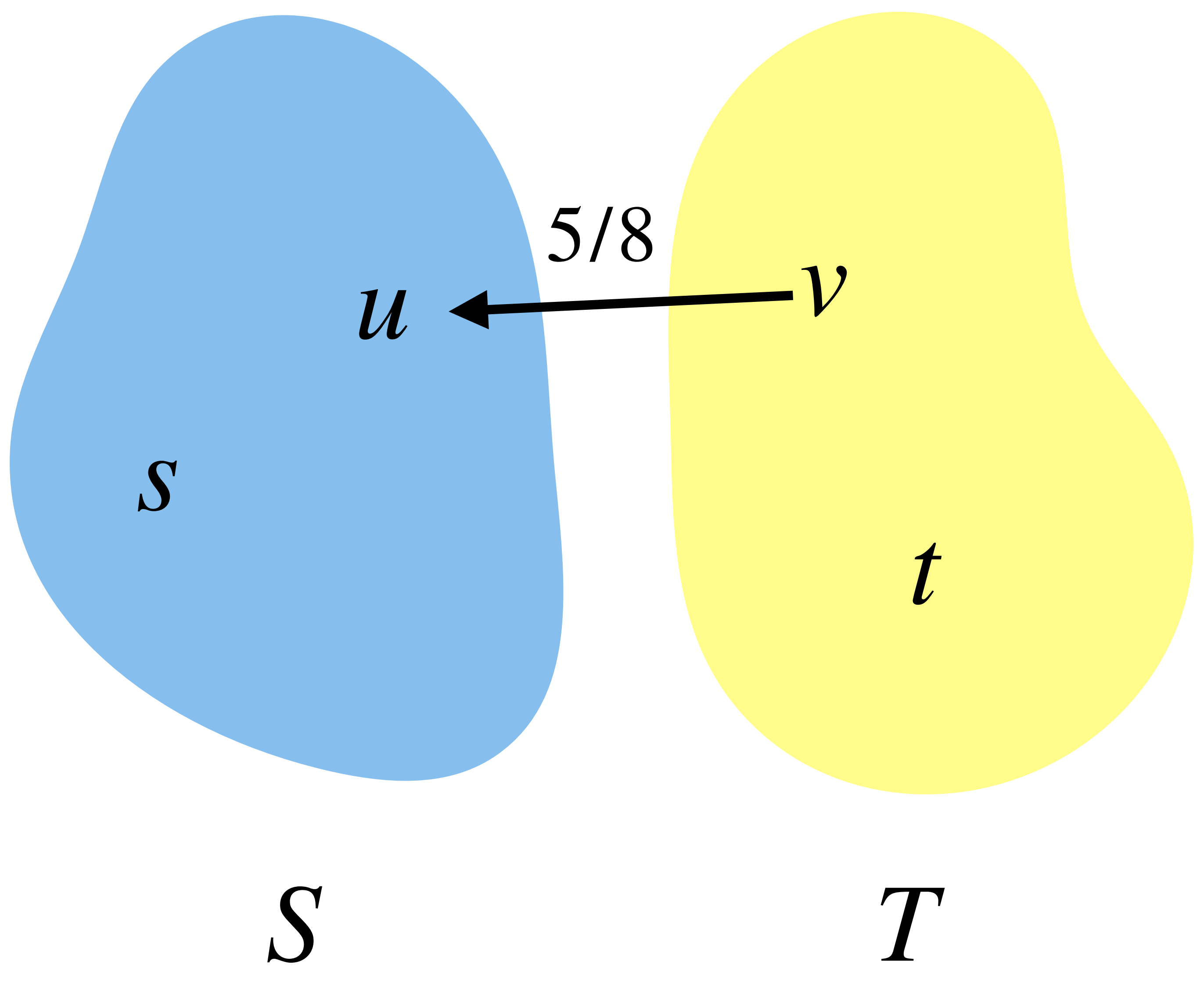

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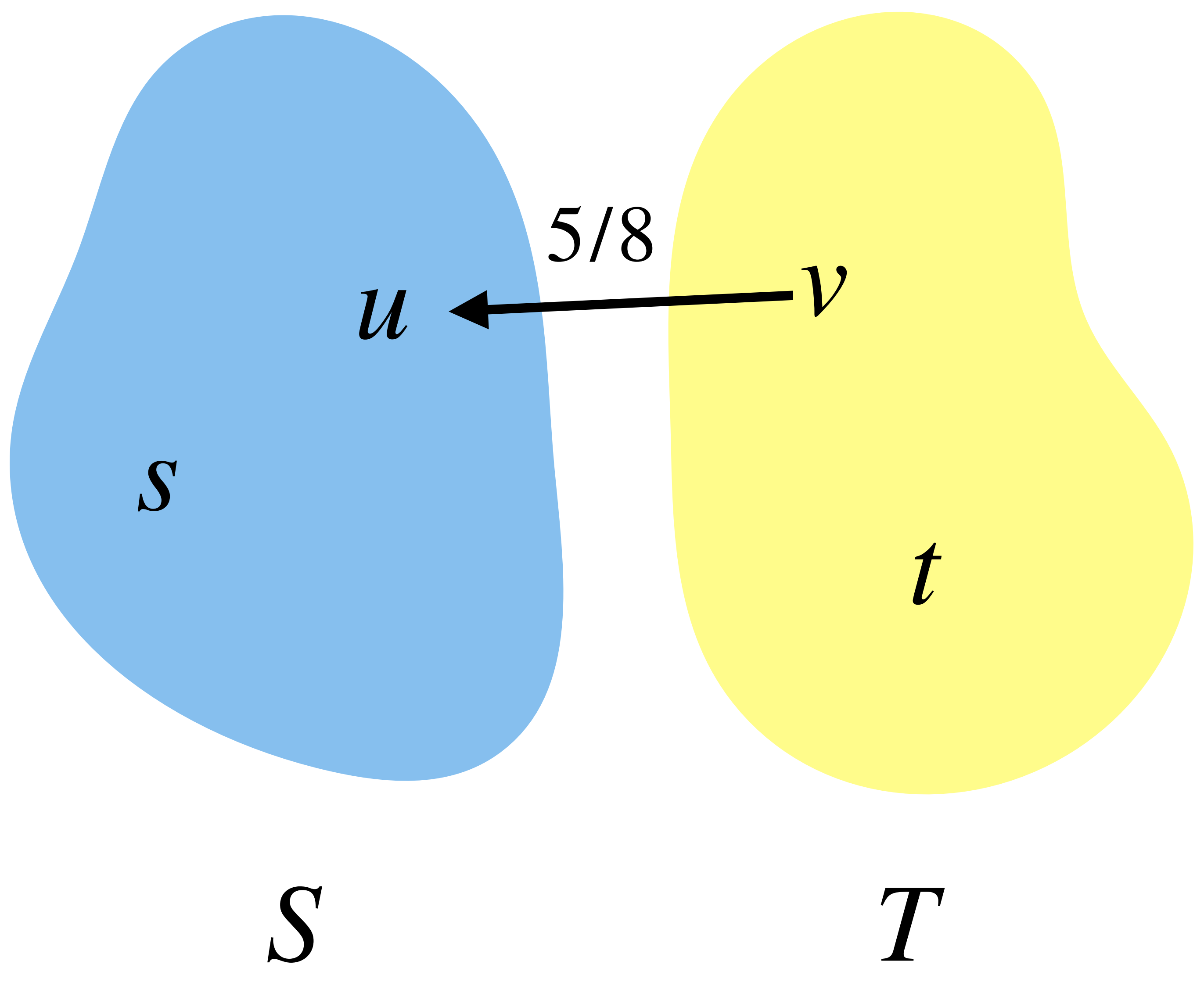

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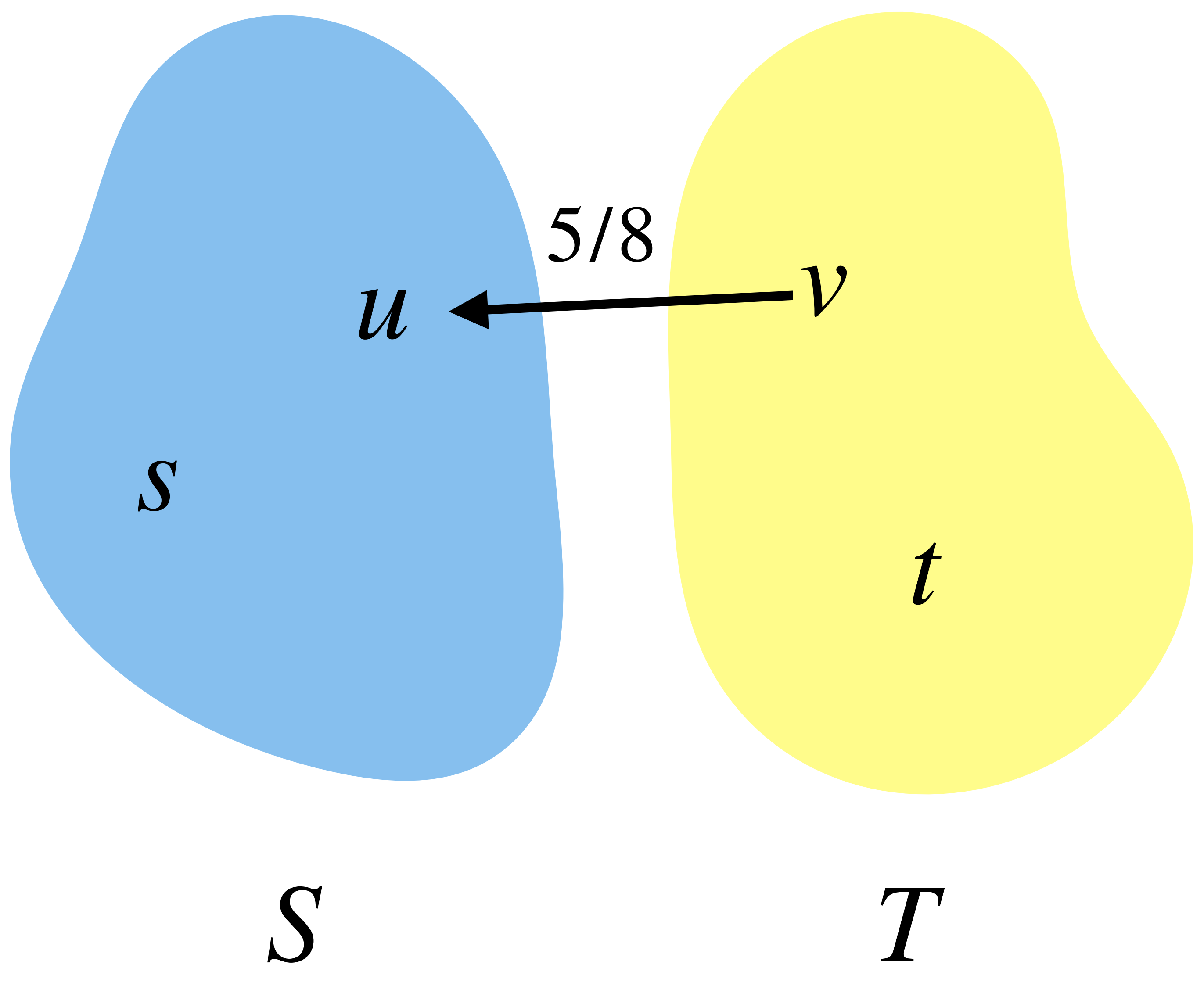

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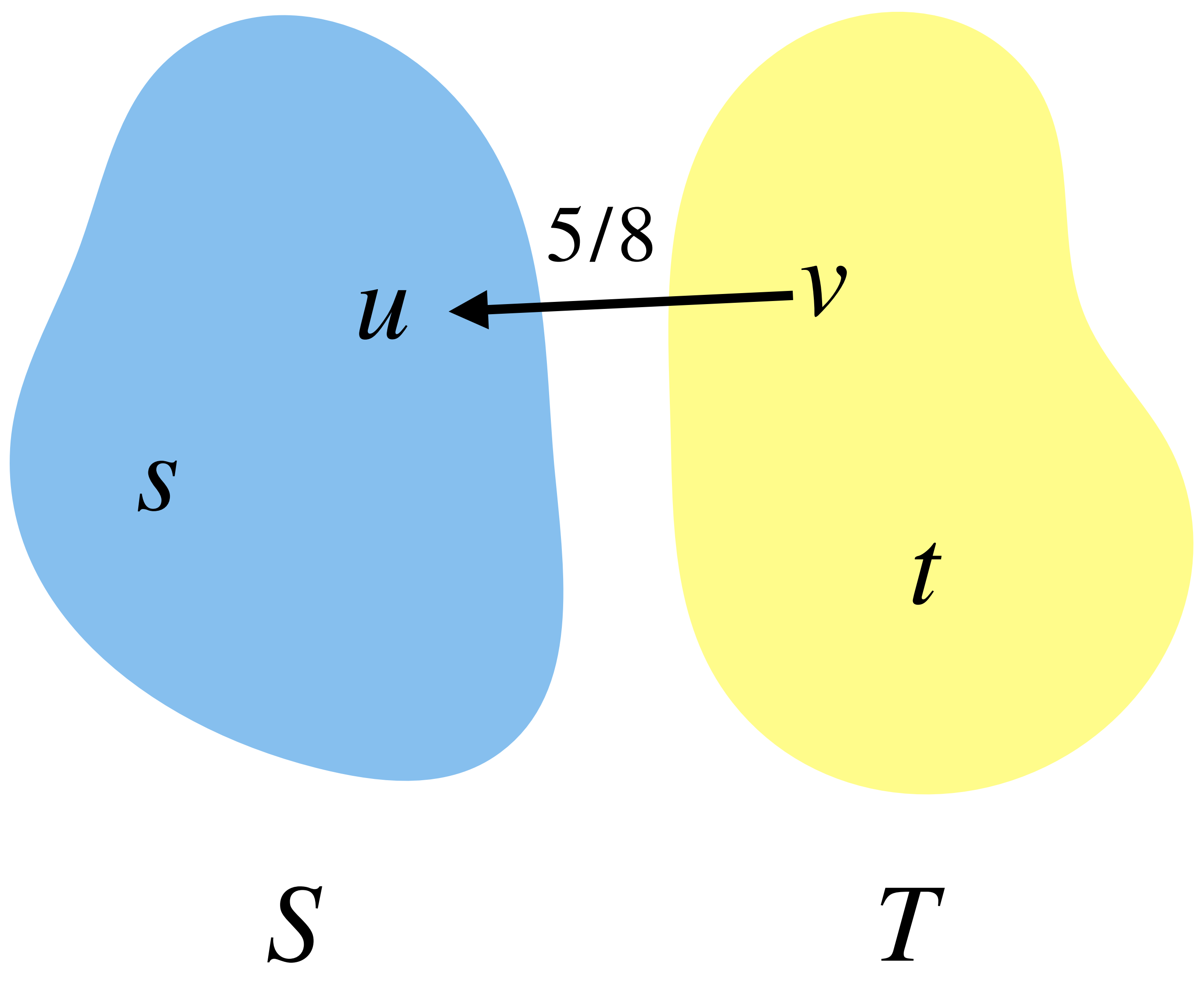

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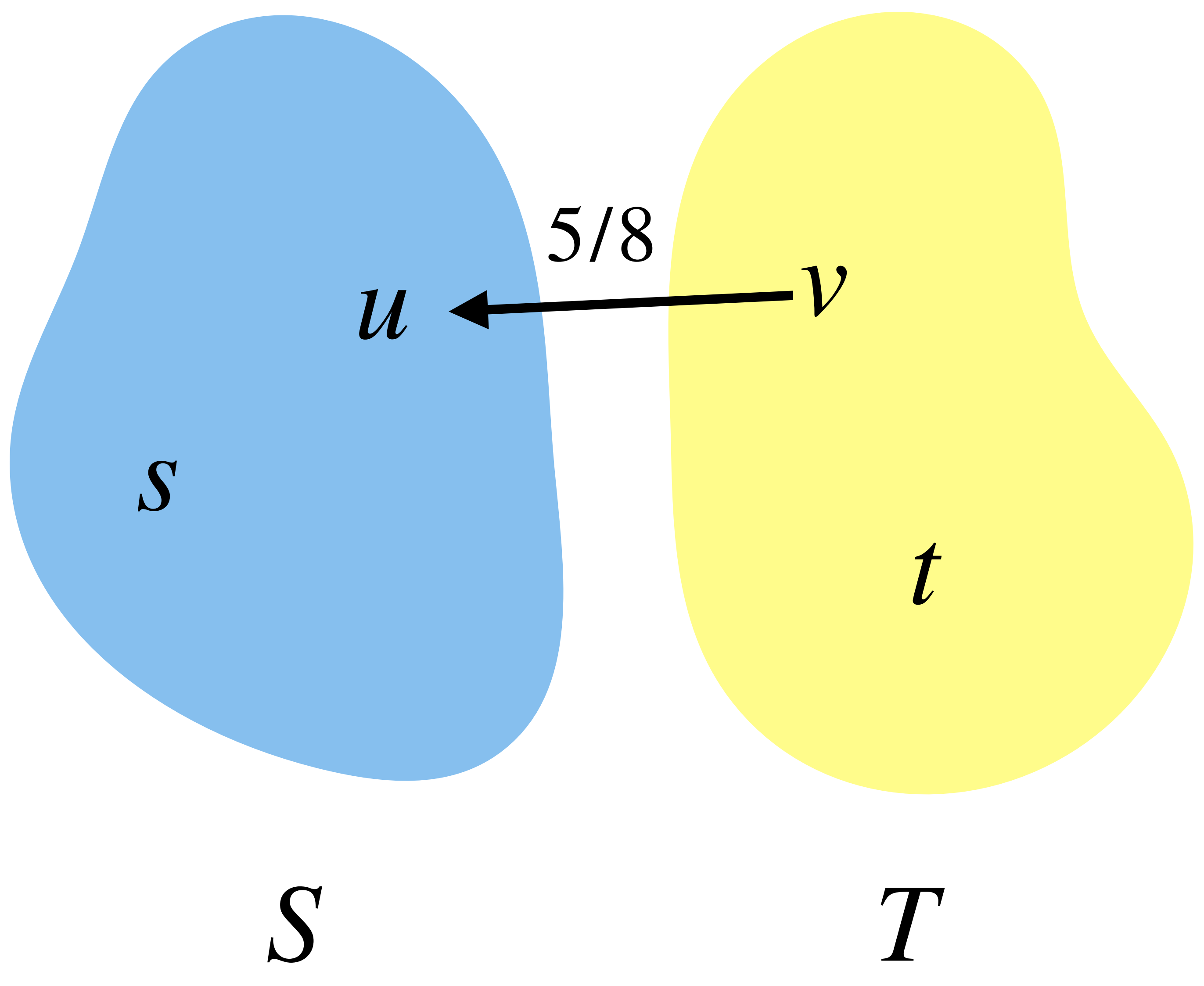

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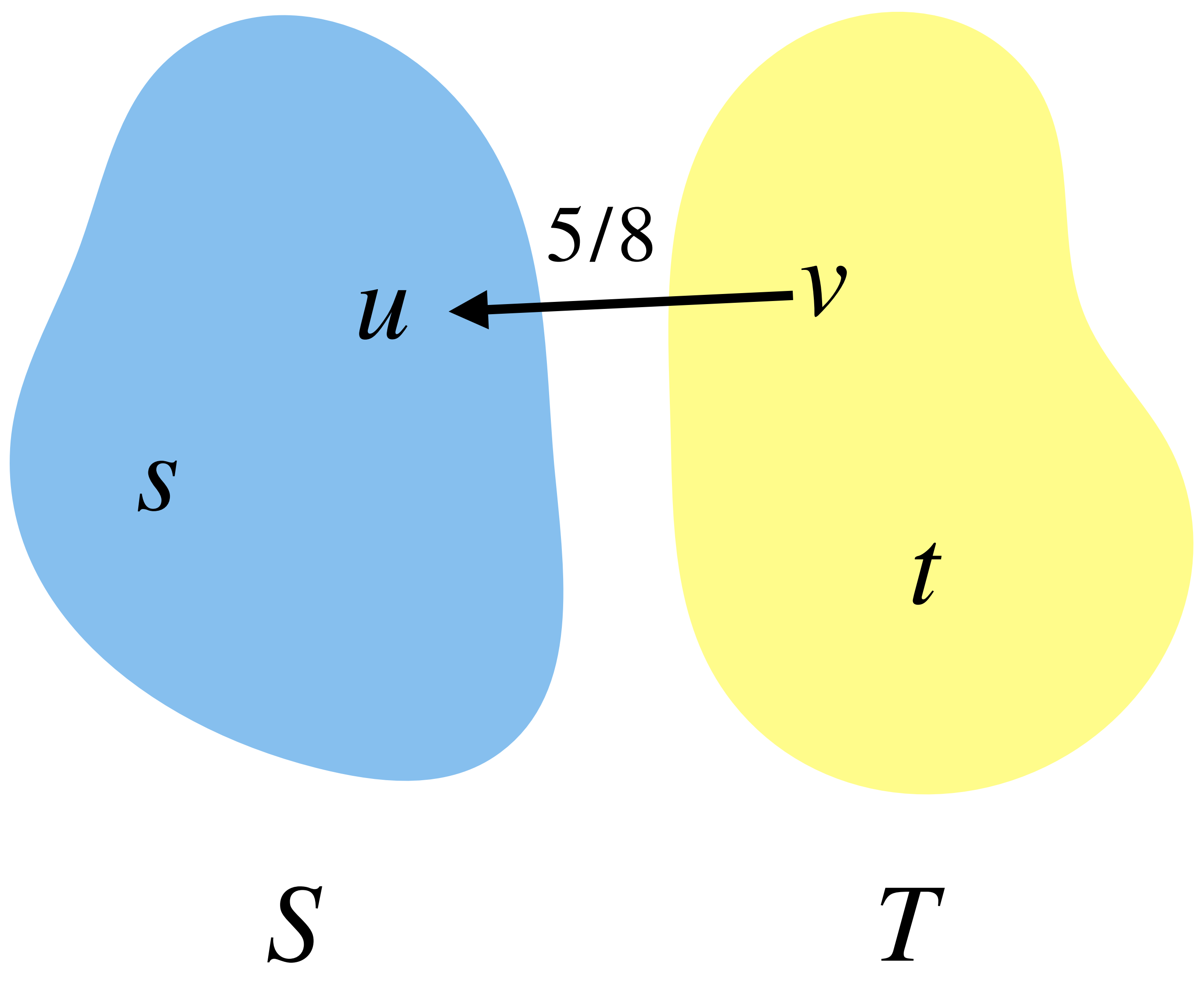

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